1 Sheet 4

1.1 Excercise 1

Find an example of an element of $SL_2(\mathbb{R})$ which is not in the image of the exponential map.

Solution. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \in SL_2(\mathbb{R}), \tag{1}$$

and suppose if possible that there exists $X \in \mathfrak{sl}_2(\mathbb{R})$ such that $e^X = A$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of X (which exists if we see X as a complex matrix) with eigenvector v. Then it is easy to see that

$$Av = e^X v = e^\lambda v.$$

This implies that $e^\lambda = -1$ and $v \in \mathbb{C}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then we find

$$X = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}.$$

Since $\lambda \notin \mathbb{R}$, this leads to a contraddiction.

1.2 Excercise 2

Show that a compact, connected complex Lie group G must be abelian, by considering its adjoint representation. Then invoke the last Exercise sheet to conclude that such a group must be of the form $\mathfrak{g}/\Gamma$ for a discrete group Γ.

Solution. The adjoint representation

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C}),\ g \mapsto \mathrm{Ad}_{g,*}$$

is a holomorphic map from the compact connected complex manifold G to an affine space, so it is constant. In fact, the image $\mathrm{Ad}(G)$ is a connected, compact complex submanifold of an affine space. The coordinate functions

$$x_i\colon\ \mathrm{Ad}(G) \to \mathbb{C}$$

must be constant by the maximum modulus theorem.[1]
In particular $\mathrm{Ad}_{g,*} = \mathrm{Id}$ for all g, and since G is connected, Theorem 4 (a) in the lectures notes tells us that the conjugation pap

$$G \longrightarrow G,\ h \mapsto ghg^{-1}$$

is the identity. Excercise 2 of excercise sheet 3 implies that $G \cong \mathfrak{g}/\Gamma$ for a discrete additive subgroup $\Gamma \subset \mathfrak{g}$.

1.3 Excercise 3

Let G be a simply connected complex Lie group, let $\mathfrak{g} = \mathrm{Lie}(G)$ and let be a real form of $\mathfrak{g}$. Show that the map

$$\mathfrak{g} \to \mathfrak{g}, \quad x + iy \mapsto x - iy$$

[1]This says that a nonconstant holomorphic function does not have local maximum.

for all $x, y \in$ can be lifted to a real Lie group automorphism $\theta : G \to G$. If we define

$$K = G^\theta = \left\{ g \in G \middle| \theta(g) = g \right\}$$

then show that K is a real Lie group with Lie algebra $\mathfrak{k}$.

Solution. The conjugation map

$$c\colon \mathfrak{g} \to \mathfrak{g}, \quad x + iy \mapsto x - iy \tag{2}$$

is a Lie algebra homomorphism and the Lie group G is simply connected. Then, Theorem 4 (b) in the lecture notes yields a lift $\theta : G \to G$. The subset $K = G^\theta$ is the kernel of a smooth automorphism of G, so it is a closed subgroup of G, By the closed subgroup theorem[2], we conclude that K is an embedded Lie group, with the structure induced by G. Then the Lie algebra $\mathrm{Lie}(K)$ is a Lie subalgebra of $\mathfrak{g}$ which we now identify. Let $X \in \mathrm{Lie}(K)$ and let $\gamma_X(t) = \exp(tX)$ be the associated (real) one parameter subgroup. Since $\gamma_X(t) \in K$ for all $t \in \mathbb{R}$, we have

$$\theta(\gamma_X(t)) = \gamma_X(t)$$

for all t. Taking the derivative in $t = 0$ yields

$$X = c(X),$$

where we used the fact that the conjugation (2) is the differential of θ at the identity $e \in G$. This implies that $\mathrm{Lie}(K)$ is isomorphic to a copy of $\mathfrak{k}$ inside $\mathfrak{g}$.

1.4 Excercise 4

Find explicit Lie algebra isomorphisms:

- $\mathfrak{so}_{3,\mathbb{C}} \cong \mathfrak{sl}_{2,\mathbb{C}}$
- $\mathfrak{so}_{4,\mathbb{R}} \cong \mathfrak{so}_{3,\mathbb{R}} \oplus \mathfrak{so}_{3,\mathbb{R}}$
- $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{1,3}$ (as real Lie algebras), where the Lorentz Lie algebra is

  $$\mathfrak{so}_{1,3} = \left\{ X \in \mathrm{Mat}_{4\times 4}(\mathbb{R}) \middle| X^T\eta + \eta X = 0 \right\}$$

  with $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

Solution. We give the requested isomorphisms by providing basis of the involved Lie algebras, which satisfy the same commutation relations.

- The matrices

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

  form a basis of $\mathfrak{so}_{3,\mathbb{C}}$. The following holds

  $$[A, B] = C, \; [B, C] = A, \; [C, A] = B. \tag{3}$$

  The matrices

[2]This is a big result that is not proven in the course. It says that a closed subgroup is an embedded Lie group with the induced smooth structure.

$$I = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, J = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, K = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

form a basis of $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}_{2,\mathbb{C}}$ such that

$$[I, J] = K,\ [J, K] = I.\ [K, I] = J.$$

- The Lie algebra $\mathfrak{so}_{4,\mathbb{R}}$ contains two subalgebras, both of which are isomorphic to $\mathfrak{so}_{3,\mathbb{R}}$, and which intersect trivially. One is generated by

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and another one by

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which satisfy the relations (3).

- Elements in the Lie algebra $\mathfrak{sl}_{1,3}$ are of the form

$$X = \begin{bmatrix} 0 & x & y & z \\ x & 0 & a & b \\ y & -a & 0 & c \\ z & -b & -c & 0 \end{bmatrix}.$$

Thus the linear subspace generated (over $\mathbb{R}$) by the matrices

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

is a subalgebra of $\mathfrak{sl}_{1,3}$ isomorphic to $\mathfrak{so}_{3,\mathbb{R}}$. As a vector subspace, this subalgebra has a complement generated by

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$[X, Y] = A', [X, Z] = B', [Y, Z] = C'.$$

This gives

$$\mathfrak{sl}_{1,3} \cong \mathfrak{so}_{3,\mathbb{R}} + i\mathfrak{so}_{3,\mathbb{R}} \cong \mathfrak{so}_{3,\mathbb{C}} \cong \mathfrak{sl}_{2,\mathbb{C}}$$

as real Lie algebras. Explicitely, the isomorphism between $\mathfrak{sl}_{1,3}$ and $\mathfrak{sl}_{2,\mathbb{C}}$ is given by

$$A' \mapsto I,\ B' \mapsto J,\ C' \mapsto K, X \mapsto iK, Y \mapsto iJ, Z \mapsto iI.$$